# MATHEMATICS AND STATISTICS FOR TECHNOLOGISTS

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CHEMICAL PUBLISHING CO., INC
212 Fifth Ave., New York, N.Y. 10010

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ISBN: 978-0-8206-0140-3

Chemical Publishing Company:
www.chemical-publishing.com
www.chemicalpublishing.net

First American edition:

© **Chemical Publishing Company, Inc. –** New York, 1967

Second Impression:

**Chemical Publishing Company, Inc. -** 2013

Printed in the United States of America

# PREFACE

A brief note of explanation concerning the origin of this book might prove of interest to the reader. Initially, it was envisaged as one volume in a series dealing with various aspects of physical processes in the chemical industry and it was intended that it should cover the mathematical techniques applied in the companion volumes. During the course of writing, however, it became increasingly evident that the contents could, with little modification, be of interest to a considerably wider readership than that for which the series was intended. It was therefore decided to publish the work in its own right.

The aim of the authors throughout has been to produce a text of wide range from which the student could derive the maximum benefit with the minimum of assistance from other sources. With this aim in view, the book starts with a revision course in basic algebra, geometry, and trigonometry, and subsequent chapters range over a wide field of mathematical techniques and their applications. Although the chapters are arranged in logical sequence, many are virtually self-contained and may be read in isolation. It is hoped that the inclusion of a large number of worked examples will materially assist the reader who is attempting to teach himself.

We should like to thank Professor J. C. Robb of Birmingham University for his helpful suggestions and Mrs. C. J. Anson for typing the manuscript.

December, 1965

H. G. Cuming
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# REVIEW OF ELEMENTARY ALGEBRA

## Indices

Algebraic notation is essentially a form of technical shorthand designed to express statements of fact in a concise and precise form. One of the earliest notations introduced to the subject concerns continued addition. Thus the expression $x+x$ is written $2x$, $x+x+x$ is written $3x$, and so on. The numbers 2 and 3 in these instances are termed 'coefficients'.

A logical development from continued addition is continued multiplication, i.e. expressions such as $x \times x$, $x \times x \times x$, etc. We write $x \times x$ as $x^2$, $x \times x \times x$ as $x^3$, and so on, the numbers 2 and 3 in these cases being termed 'indices' or 'exponents'. (It should be realized that in the same way that $x$ is a shortened form of $1x$, $x$ is also a shortened form of $x^1$, the coefficient 1 and the index 1 both being conventionally omitted.) Quite generally, if $n$ is any positive integer, $x^n$ simply means the product of $n$ $x$'s. The laws of indices follow directly from this definition: we begin by considering positive integral indices.

*Multiplication law*

This states that

$$x^p \times x^q = x^{p+q},$$

i.e. to multiply, add indices. The result is obtained as follows.

Since $\quad x^p = x \times x \times x \times \ldots \times x,\ p$ times

and $\quad x^q = x \times x \times x \times \ldots \times x,\ q$ times

therefore $\quad x^p \times x^q = (x \times x \ldots \times x,\ p \text{ times}) \times (x \times x \ldots \times x,\ q \text{ times})$

$$= x \times x \times x \times \ldots \times x,\ (p+q) \text{ times}$$

$$= x^{p+q} \qquad \ldots (1.1)$$

*Examples* $\quad x^4 \times x^3 = x^7$; $x \times x^n = x^{n+1}$; $3x^2 \times 5x^4 = 15x^6$

*Division law*

This states that

$$x^p \div x^q = x^{p-q}$$

i.e. to divide, subtract indices. To prove the result, we note that

$$\frac{x^p}{x^q} = \frac{x \times x \times x \times \ldots \times x,\ p \text{ times}}{x \times x \times x \times \ldots \times x,\ q \text{ times}}$$

$$= \frac{(x \times x \ldots x, (p-q) \text{ times}) \times (x \times x \ldots x, q \text{ times})}{x \times x \times x \ldots \times x,\ q \text{ times}}$$

where the total of $p$ $x$'s in the numerator have been put into two groups, one containing $q$ and the other the remaining $(p-q)$. The group of $q$ $x$'s in the numerator cancels with the identical group in the denominator, giving

$$\frac{x^p}{x^q} = x \times x \times x \times \ldots \times x,\ (p-q) \text{ times}$$

$$= x^{p-q} \qquad \ldots (1.2)$$

*Examples* $x^5 \div x^2 = x^3$; $x^n \div x = x^{n-1}$; $21x^8 \div 7x^2 = 3x^6$

*Power law*

This states that

$$(x^p)^q = x^{pq}$$

i.e. to raise to a power, multiply indices. Starting from the definition

$$x^p = x \times x \times x \times \ldots \times x,\ p \text{ times}$$

$$(x^p)^q = (x \times x \ldots \times x,\ p \text{ times}) \times (x \times x \times x \ldots \times x,\ p \text{ times}) \times \ldots$$
$$\times (x \times x \ldots \times x,\ p \text{ times})$$

the number of bracketed groups being $q$,

$$= x \times x \times x \times \ldots \times x,\ pq \text{ times}$$

$$= x^{pq} \qquad \ldots (1.3)$$

It should be carefully noted that there is no contradiction between the multiplication and power laws; the latter is, in fact, the result of a repeated application of the former.

*Examples* $(2^2)^3 = 2^6 = 64$; $(3x^2)^4 = 81x^8$

*The significance of* $x^0$

Applying the division law to the problem of dividing $x^p$ by $x^p$ we obtain, subtracting indices,

$$x^p \div x^p = x^{p-p} = x^0$$

It is clear, however, that the result of dividing $x^p$ by itself must be 1 (except in the case when $x = 0$, when the quotient takes the form of a limit: this is discussed in Chapter 7). Thus

$$x^0 = 1 \qquad \ldots (1.4)$$

for all values of $x$ except zero. Thus

$$1^0 = 1,\ 100^0 = 1,\ (-1000)^0 = 1$$

*Negative integral indices*

Having established the significance of positive integral indices and the index zero, we now examine the possibility of attaching a meaning to negative integral indices, e.g. to such expressions as $x^{-2}$, $x^{-4}$, $x^{-n}$. If we apply the multiplication law to the product of $x^p$ and $x^{-p}$,

$$\begin{aligned} x^p \times x^{-p} &= x^{p+(-p)} \\ &= x^0 \\ &= 1 \\ x^{-p} &= \frac{1}{x^p} \end{aligned} \qquad \ldots (1.5)$$

i.e. $x^{-p}$ is simply the reciprocal of $x^p$.

*Examples* $10^{-1} = \dfrac{1}{10}$; $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100}$;

$$(-4)^{-3} = \frac{1}{(-4)^3} = -\frac{1}{64}$$

*Fractional indices*

Having covered all cases where the index is integral (positive, zero, and negative), we conclude by attaching a meaning to fractional indices of the form $p/q$, where the numbers $p$ and $q$ are integral. Consider the product of $x^{1/q}$ with itself $q$ times; using the power law

$$\begin{aligned} x^{1/q} \times x^{1/q} \times \ldots \times x^{1/q},\ q \text{ times} &= (x^{1/q})^q \\ &= x^1 \\ &= x \end{aligned}$$

It follows that $x^{1/q}$ is that quantity which, multiplied by itself $q$ times, gives $x$, i.e. $x^{1/q}$ is the $q$th root of $x$. Thus

$$x^{1/q} = \sqrt[q]{x} \qquad \ldots (1.6)$$

Therefore $x^{1/2} = \sqrt{x}$; $x^{1/3} = \sqrt[3]{x}$; $x^{1/4} = \sqrt[4]{x}$, and so on. It follows that

$$\left.\begin{aligned} & x^{p/q} = (x^{1/q})^p = (\sqrt[q]{x})^p \\ \text{and} \quad & x^{p/q} = (x^p)^{1/q} = \sqrt[q]{(x^p)} \end{aligned}\right\} \qquad \ldots (1.7)$$

*Example* Evaluate $(27)^{2/3}$.

There are three ways of proceeding. We could write

$$(27)^{2/3} = (\sqrt[3]{27})^2 = (3)^2 = 9$$

Alternatively

$$(27)^{2/3} = \sqrt[3]{(27)^2} = \sqrt[3]{729} = 9$$

Clearly, the second method involves more arithmetic. The best method is to work throughout in indices, thus

$$(27)^{2/3} = (3^3)^{2/3} = 3^2 = 9$$

*Exercise* 1(*a*). Evaluate the following expressions:

(1) $\frac{5}{6}x^3 \times \frac{3}{25}x^4 \div 15x^2$ (2) $\frac{2^4 \times 3^6}{3^8 \times 2}$

(3) $10^{-5}$ (4) $(-\frac{1}{2})^{-3}$

(5) $(49)^{-0\cdot 5}$ (6) $4^3 \times 4^{-3/2}$

(7) $\left(\frac{3^4}{2^6}\right)^{-1/2}$ (8) $(100)^{5/2}$

(9) $(\frac{343}{64})^{-2/3}$ (10) $\sqrt[3]{(8^{-1})}$

(11) $(16x^2)^{3/4} \times (125x^3)^{-2/3}$ (12) $\frac{\sqrt[4]{(x^{2\cdot 8})}}{\sqrt{(x^{1\cdot 6})}}$

## Logarithms

Having defined a direct algebraic operation we may then go on to devise the reverse, or inverse, operation. Such operations are analagous to forward and reverse gearing, i.e. the effect of the two operations carried out one after the other being to leave the position unaltered. One example of a pair of mutually inverse operations is multiplication and division by the same number; another is raising to the *n*th power and taking the *n*th root. Since the effect of the inverse operation is to nullify that of the direct operation, it follows that inverse operations provide a means of solving equations. For example,

$$y = 3x$$

and

$$x = \frac{y}{3}$$

are equivalent statements and each may be regarded as the solution of the other: starting with $x$, we multiply by 3 to obtain $y$; starting with $y$, we divide by 3 to obtain $x$.

Consider now the indicial, or exponential, relationship $y = a^x$, i.e. starting with $x$ we raise $a$ to the power $x$ to obtain $y$. Suppose it is required to solve this equation to express $x$ in terms of $y$; clearly we must apply the operation which is the inverse of the exponential operation. This inverse operation is known as the 'logarithmic' operation and is defined as follows. If $y = a^x$, then the exponent $x$ is the logarithm of $y$ to base $a$ and we write $x = \log_a y$. Put in a slightly different way, this definition states that the logarithm of a number ($y$) to a given base ($a$) is the power ($x$) to which the base must be raised to equal that number. The important point to note is that

$$\left.\begin{aligned} y &= a^x \\ \text{and} \qquad x &= \log_a y \end{aligned}\right\} \qquad \ldots (1.8)$$

are equivalent statements differing only in form, and that each may be regarded as the solution of the other. The first is known as the 'exponential' form and the second as the 'logarithmic' form.

*Examples* (1) Since $64 = 4^3$ (exponential form), then $3 = \log_4 64$ (logarithmic form).

(2) Since $\dfrac{1}{100} = 10^{-2}$ (exponential form), then $-2 = \log_{10}\left(\dfrac{1}{100}\right)$ (logarithmic form).

Since to every exponential relationship there corresponds an inverse logarithmic form, it follows that the laws of indices may be expressed as logarithmic laws. These we now proceed to derive.

*Multiplication law*

This states that

$$\log_a(xy) = \log_a x + \log_a y$$

From the definition, if $\log_a x = p$, then $x = a^p$,

and if $\log_a y = q$, then $y = a^q$; therefore

$$xy = a^p \times a^q = a^{p+q}$$

Therefore

$$\begin{aligned} \log_a(xy) &= p+q \\ &= \log_a x + \log_a y \end{aligned} \qquad \ldots (1.9)$$

i.e. the logarithm of a product is equal to the sum of the logarithms of the separate factors.

*Division law*

This states that

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

With the notation of the preceding section,

$$\frac{x}{y} = \frac{a^p}{a^q} = a^{p-q}$$

Thus

$$\begin{aligned} \log_a\left(\frac{x}{y}\right) &= p-q \\ &= \log_a x - \log_a y \end{aligned} \qquad \ldots (1.10)$$

i.e. the logarithm of a quotient is equal to the logarithm of the numerator minus the logarithm of the denominator.

*Power law*

This states that

$$\log_a(x^q) = q \log_a x$$

From the definition, if $\log_a x = p$, then $x = a^p$, therefore

$$x^q = (a^p)^q = a^{pq}$$

Therefore $$\log_a(x^q) = qp = q \log_a x \qquad \ldots (1.11)$$

*Examples* (1) $\log_a 6 = \log_a 2 + \log_a 3$

$$(2)\ \log_{10}\left(\frac{14}{15}\right) = \log_{10}\left(\frac{2\times 7}{3\times 5}\right)$$

$$= \log_{10} 2 + \log_{10} 7 - \log_{10} 3 - \log_{10} 5$$

$$(3)\ \log_4(\sqrt[5]{2}) = \log_4(2^{1/5})$$

$$= \log_4(4^{1/10})$$

$$= \tfrac{1}{10}$$

*Change of base of logarithms*

The logarithm of a given number depends upon the choice of base; however, given the value of the logarithm to one base, we may easily deduce the value corresponding to another base. Let the two bases be $a$ and $b$. If

$$y = \log_a x$$

then

$$x = a^y$$

and if

$$z = \log_b x$$

then

$$x = b^z$$

$$\therefore\ b^z = a^y$$

Taking logarithms to base $b$ of both sides

$$z \log_b b = y \log_b a$$

i.e. $z = y \log_b a$ (since $\log_b b = 1$)

i.e. $\log_b x = (\log_a x)\times(\log_b a) \qquad \ldots (1.12)$

*Example* $\log_{\sqrt{2}} 16 = (\log_2 16)\times(\log_{\sqrt{2}} 2)$

$$= 4\times 2$$

$$= 8$$

*Practical uses of logarithms*

Logarithms have an immediate application to evaluating products, quotients, and powers, for, by using the laws we have derived, these operations may be replaced by the simpler processes of addition, subtraction, and multiplication, respectively. In order to take advantage of this simplification we must construct a numerical table of logarithms from which may be read the logarithm of any number. Before constructing such a table, we must first decide upon a numerical value for the base. Although the laws of logarithms hold for any value of the base, in practice we virtually restrict ourselves to one of two values, i.e. either 10 or a number denoted by $e$ ($e$ is a number which arises in more advanced theoretical work: it is called the 'exponential number' and is discussed more fully in Chapter 8). Logarithms to base 10 are called 'common' logarithms and are mostly used for numerical calculation; logarithms to base $e$ are called 'natural' or 'Naperian' logarithms, and are used in theoretical work in which they are of more convenience than common logarithms. In this chapter we shall restrict our attention to common logarithms.

If we put $x = 10^y$ and let $y$ take integral values between $-4$ and $+4$, we obtain the following Table.

| $y$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | 0·0001 | 0·001 | 0·01 | 0·1 | 1·0 | 10 | 100 | 1,000 | 10,000 |

Since $x = 10^y$, then $y = \log_{10} x$, and the Table may be rewritten in the following form.

| $x$ | 0·0001 | 0·001 | 0·01 | 0·1 | 1·0 | 10 | 100 | 1,000 | 10,000 |
|---|---|---|---|---|---|---|---|---|---|
| $\log_{10} x$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 |

It now remains to fill in the gaps between the values of $x$ chosen in the Table. The details of the actual calculation of these depends upon more advanced work (in fact, using the Taylor series, discussed in Chapter 9). It is clear, however, that in general the logarithm will consist of two parts, an integer and a decimal less than 1. (For example, since $\log_{10} 10 = 1$ and $\log_{10} 100 = 2$, $\log_{10} 20$ will lie between 1 and 2.) The integral part of the logarithm is called the 'characteristic' and the decimal part the 'mantissa.'

The advantage of choosing 10 as the base of logarithms will now be made clear. Suppose that the logarithms of numbers between 1 and 10 have been tabulated and that $x$ is any such number. Then $\log_{10} x$ lies between 0 and 1 and, using the division law,

$$\left.\begin{aligned}
\log_{10}\left(\frac{x}{100}\right) &= \log_{10}x-\log_{10}100 = -2+\log_{10}x\\
\log_{10}\left(\frac{x}{10}\right) &= \log_{10}x-\log_{10}10 \;\;\, = -1+\log_{10}x\\
\log_{10}x &= \log_{10}x-\log_{10}1 \;\;\;\, = \;\;\, 0+\log_{10}x\\
\log_{10}(10x) &= \log_{10}x+\log_{10}10 \;\;\, = \;\;\, 1+\log_{10}x\\
\log_{10}(100x) &= \log_{10}x+\log_{10}100 = \;\;\, 2+\log_{10}x
\end{aligned}\right\} \quad \ldots (1.13)$$

and so on, in which the first column on the right-hand side represents the characteristics and the second column the mantissae. We observe that the mantissae are all equal and that the logarithms differ only in their characteristics, which increase by 1 each time the number is multiplied by a factor of 10. In practice, therefore, we need tabulate only the logarithms of numbers between 1 and 10; this is sufficient to determine the mantissa of any number and the characteristic is determined independently. This is equivalent to saying that given a number in decimal form consisting of some sequence of digits, the mantissa is determined completely by that sequence irrespective of the position of the decimal point, the latter serving only to determine the value of the characteristic. For example, putting $x = 2{\cdot}147$ in equations (1.13) and given that $\log_{10} 2{\cdot}147 = 0{\cdot}3318$, we have

$$\left.\begin{aligned}
\log_{10}0{\cdot}02147 &= -2+0{\cdot}3318\\
\log_{10}0{\cdot}2147 &= -1+0{\cdot}3318\\
\log_{10}2{\cdot}147 &= \;\;\, 0+0{\cdot}3318\\
\log_{10}21{\cdot}47 &= \;\;\, 1+0{\cdot}3318\\
\log_{10}214{\cdot}7 &= \;\;\, 2+0{\cdot}3318
\end{aligned}\right\} \quad \ldots (1.14)$$

and so on. These results may be generalized in the following rules.

(1) For a number greater than 1, the characteristic is positive and one less than the number of digits before the decimal point.

(2) For a number less than 1, the characteristic is negative and *numerically* one greater than the number of zeros immediately following the decimal point. In this case we adopt a special notation. For example, $\log_{10} 0{\cdot}2147 = -1+0{\cdot}3318$ and clearly we cannot write this as $-1{\cdot}3318$, since the latter expression means $-1-0{\cdot}3318$. Instead, we write $\log_{10} 0{\cdot}2147 = \bar{1}{\cdot}3318$, the bar over the 1 denoting that it alone is to be considered negative, $0{\cdot}3318$ being taken positive. If we use this notation we can rewrite equations (1.14) as

$$\left.\begin{aligned}
\log_{10}0{\cdot}02147 &= \bar{2}{\cdot}3318\\
\log_{10}0{\cdot}2147 &= \bar{1}{\cdot}3318\\
\log_{10}2{\cdot}147 &= 0{\cdot}3318\\
\log_{10}21{\cdot}47 &= 1{\cdot}3318\\
\log_{10}214{\cdot}7 &= 2{\cdot}3318
\end{aligned}\right\} \quad \ldots (1.15)$$

and so on. The reader should familiarize himself with performing the basic operations of addition, subtraction, multiplication, and division using the bar notation. These are carried out exactly as in ordinary arithmetic as far as the mantissae are concerned; we simply remember that $\bar{6}$, for example, means $-6$ when we come to the characteristics.

*Examples* (1) Addition:

$$\begin{array}{l} \bar{2}\cdot 3146+ \\ \bar{4}\cdot 8249 \\ \hline \bar{5}\cdot 1395 \\ \hline \end{array}$$

since $\bar{2}+\bar{4} = -2-4 = -6 = \bar{6}$, and the $+1$ to be carried over from adding the mantissae makes this up to $\bar{5}$.

(2) Subtraction:

$$\begin{array}{l} \bar{2}\cdot 3146- \\ \bar{4}\cdot 8249 \\ \hline 1\cdot 4897 \\ \hline \end{array}$$

since $\bar{2}-\bar{4} = -2-(-4) = 2$, and 1 must be subtracted from this due to the carry over from subtracting the mantissae.

(3) Multiplication:

$$\begin{aligned} 3\times\bar{2}\cdot 5432 &= (3\times\bar{2})+(3\times 0\cdot 5432) \\ &= \bar{6}+1\cdot 6296 \\ &= \bar{5}\cdot 6296 \end{aligned}$$

(4) Division:

$$\begin{aligned} \frac{\bar{4}\cdot 6328}{2} &= \frac{\bar{4}}{2}+\frac{0\cdot 6328}{2} \\ &= \bar{2}+0\cdot 3164 \\ &= \bar{2}\cdot 3164 \end{aligned}$$

If the characteristic is not exactly divisible by the divisor, we proceed as follows:

$$\begin{aligned} \frac{\bar{5}\cdot 6328}{2} &= \frac{\bar{5}+0\cdot 6328}{2} \\ &= \frac{\bar{6}+1\cdot 6328}{2} \\ &= \frac{\bar{6}}{2}+\frac{1\cdot 6328}{2} \\ &= \bar{3}+0\cdot 8164 \\ &= \bar{3}\cdot 8164 \end{aligned}$$

i.e. since $\bar{5}$ is not exactly divisible by 2, we increase the number under the bar to the next highest integer which is divisible by 2. Since this is 6, we have *subtracted* 1 from the number, and this is compensated for by adding 1 to the mantissa.

We come now to discuss the method of reading mantissae from a table of logarithms. The following is an extract from a set of four-figure common logarithms covering the range 6500 – 6999.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | *Differences* 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 65 | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 66 | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8248 | 8254 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 67 | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 8319 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 68 | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 69 | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |

To fix ideas, suppose it is required to find $\log_{10} 65{\cdot}37$. To begin with, we ignore the decimal point and concentrate on the sequence 6537. First locate the number 65, formed by the first two digits, in the left-hand column; then proceed horizontally to the right until the first column headed 3 is reached. This pinpoints the number 0·8149. Now proceed along the 'Differences' columns until the column headed 7 is reached; this gives the number 0·0005. The required mantissa is then $0{\cdot}8149+0{\cdot}0005 = 0{\cdot}8154$. Taking note of the position of the decimal point in the number 65·37, we observe that the characteristic is 1 and hence $\log_{10} 65{\cdot}37 = 1{\cdot}8154$.

*Examples* (1) Evaluate $\log_{10}(6{\cdot}5\times 65{\cdot}37)$:

$$\begin{aligned}\log_{10}(6{\cdot}5\times 65{\cdot}37) &= \log_{10} 6{\cdot}5+\log_{10} 65{\cdot}37\\ &= 0{\cdot}8129+1{\cdot}8154\\ &= 2{\cdot}6283\end{aligned}$$

(2) Evaluate $\log_{10}\left(\dfrac{6{\cdot}5}{65{\cdot}37}\right)$:

$$\begin{aligned}\log_{10}\left(\frac{6{\cdot}5}{65{\cdot}37}\right) &= \log_{10} 6{\cdot}5-\log_{10} 65{\cdot}37\\ &= 0{\cdot}8129-1{\cdot}8154\\ &= \bar{2}{\cdot}9975\end{aligned}$$

(3) Evaluate $\log_{10}\sqrt{(65{\cdot}37)}$:

$$\log_{10}\sqrt{(65{\cdot}37)} = \log_{10}(65{\cdot}37)^{1/2}$$

$$= \tfrac{1}{2} \log_{10} 65{\cdot}37$$
$$= \tfrac{1}{2} \times 1{\cdot}8154$$
$$= 0{\cdot}9077$$

It is clear from these examples that by using the laws of logarithms in conjunction with the table of logarithms we may easily form the logarithms of products, quotients, and powers. In order to complete the calculations, we must be able to determine the number which has a given logarithm; this is known as 'finding the antilogarithm'. Thus, if $\log_{10} 65{\cdot}37 = 1{\cdot}8154$, then $\text{antilog}_{10}\, 1{\cdot}8154 = 65{\cdot}37$. (This is another example of inverse operations.) Although there are tables of antilogarithms available from which this may be done directly, the same result may also be achieved by using the logarithm tables in reverse order. For example, to find $\text{antilog}_{10}\, 1{\cdot}8154$, first ignore the characteristic 1 and search for the mantissa 0·8154 in the body of the Table. This number does not in fact occur; the closest approximations are 0·8149 and 0·8156, corresponding to the sequences 6530 and 6540, respectively. Since 0·8149 is smaller than 0·8154 by 0·0005, we proceed horizontally to the right until the number 5 in the 'Differences' columns is reached: this corresponds to the sequence 0007 which is added to 6530 to obtain the sequence 6537. Finally, since the given characteristic is 1, there must be *two* digits before the decimal point. Therefore

$$\text{antilog}_{10}\, 1{\cdot}8154 = 65{\cdot}37$$

The method of setting out logarithmic calculations is illustrated in the following examples.

*Examples* Using logarithms, evaluate

(1) $\dfrac{4{\cdot}796 \times 2{\cdot}314}{0{\cdot}7627 \times 53{\cdot}16}$ (2) $\sqrt[3]{(0{\cdot}6148)^2}$

(1) $\log_{10} \dfrac{4{\cdot}796 \times 2{\cdot}314}{0{\cdot}7627 \times 53{\cdot}16}$

$= \log_{10} 4{\cdot}796 + \log_{10} 2{\cdot}314$

$\quad - \log_{10} 0{\cdot}7627 - \log_{10} 53{\cdot}16$

$= \bar{1}{\cdot}4371$

Therefore $\dfrac{4{\cdot}796 \times 2{\cdot}314}{0{\cdot}7627 \times 53{\cdot}16}$

$= \text{antilog}_{10}\, \bar{1}{\cdot}4371$

$= 0{\cdot}2736$

| *No.* | *Log* |
|---|---|
| 4·796 | 0·6808 + |
| 2·314 | 0·3643 |
| | 1·0451 − |
| 0·7627 | $\bar{1}{\cdot}8824$ |
| | 1·1627 − |
| 53·16 | 1·7256 |
| 0·2736 | $\bar{1}{\cdot}4371$ |

(2) $\sqrt[3]{(0{\cdot}6148)^2} = (0{\cdot}6148)^{2/3}$

Therefore
$$\log_{10}\sqrt[3]{(0{\cdot}6148)^2}$$
$$= \log_{10}(0{\cdot}6148)^{2/3}$$
$$= \tfrac{2}{3}\log_{10}0{\cdot}6148$$
$$= \bar{1}{\cdot}8592$$

| *No.* | *Log* |
| --- | --- |
| 0·6148 | $\bar{1}{\cdot}7888 \times 2$ |
| | $\bar{1}{\cdot}5776 \div 3$ |
| 0·7231 | $\bar{1}{\cdot}8592$ |

Therefore
$$\sqrt[3]{(0{\cdot}6148)^2} = \text{antilog}_{10}\,\bar{1}{\cdot}8592$$
$$= 0{\cdot}7231$$

*Exercise* 1(*b*). Evaluate, using logarithms:

(1) $76{\cdot}14 \times 0{\cdot}005823 \times 0{\cdot}01732$

(2) $\dfrac{214{\cdot}8 \times 0{\cdot}1537}{987{\cdot}6}$

(3) $\dfrac{47{\cdot}64 \times 8{\cdot}271}{563{\cdot}5 \times 0{\cdot}0157}$

(4) $\sqrt{(947)}$

(5) $\sqrt[3]{(18{\cdot}32)}$

(6) $\sqrt[5]{(472{\cdot}5)^2}$

(7) $\dfrac{(19{\cdot}31)^2 \times (5{\cdot}361)^3}{(25{\cdot}41)^3 \times (0{\cdot}1496)^4}$

(8) $(14{\cdot}27)^2 - (5{\cdot}62)^2$

(9) $\sqrt{\left(\dfrac{2{\cdot}483 \times 7{\cdot}651}{44{\cdot}62}\right)}$

(10) $\sqrt[5]{\left(\dfrac{(35{\cdot}91)^3}{(72{\cdot}14)^2}\right)}$

## GRAPHS

Mathematics is concerned with the relationships between connected quantities. For example, if $A$ denotes the area of a square of side $x$, then $A = x^2$. In this particular example, both $A$ and $x$ possess physical significance and the relationship between them can be expressed in the form of an equation. Equally, a relationship which can be expressed by an equation can also exist between abstract quantities. Another type of relationship, associated with experimental work, is that expressed in the form of a table of corresponding values. For example, during the course of a chemical reaction we may measure the concentration of a particular substance at known times and record the results (i.e. corresponding values of concentration and time) in numerical form.

The equation in the one case and the table of values in the other contain, in principle, all the information we have concerning the particular relationship in question, but the form in which this information is couched is not suited to quick and easy assimilation. It is with this object in view that we devise a pictorial representation of the relationship, known as a 'graph'.

To fix ideas, consider the relationship expressed by

$$y = 2x+1$$

We start by constructing a set of corresponding values of $x$ and $y$; putting $x = -3, -2, -1, 0, 1, 2, 3$ in turn we obtain the Table below, in which the numbers occurring in the last row are the sums of the two numbers immediately above.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
| $2x$ | $-6$ | $-4$ | $-2$ | 0 | 2 | 4 | 6 |
| $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ |
| $y$ | $-5$ | $-3$ | $-1$ | 1 | 3 | 5 | 7 |

A sheet of squared paper is now divided into four parts (quadrants) by two perpendicular lines O$x$, O$y$ (Figure 1). The lines O$x$, O$y$ are called the '$x$' and '$y$'

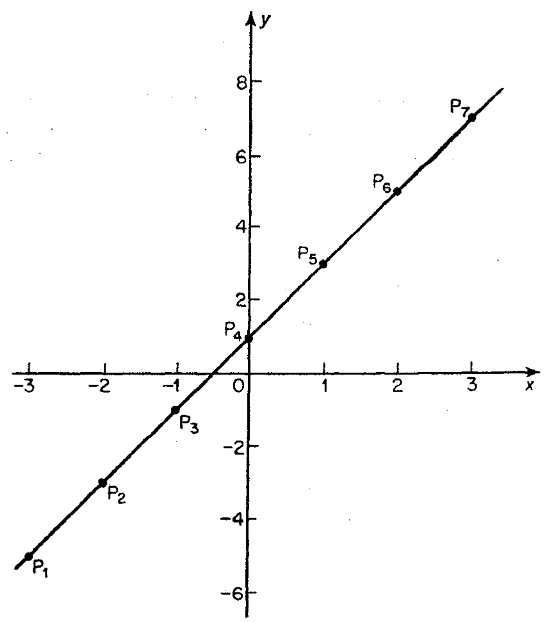


FIG. 1

axes, respectively, and their point of intersection O is called the 'origin'. Using a convenient scale which covers the range of values of $x$ and $y$ in the Table, the axes are marked off in successive units; they are positive when proceeding to the right from O along O$x$ and when proceeding upward from O along O$y$, and negative in the reverse directions. (Note that different scales may be used on the $x$ and $y$ axes, provided the *same* scale is used on both sides of *each* axis.)

The first pair of corresponding values in the Table, i.e. $x = -3, y = -5$, is now represented on the paper as a point in the following manner. Starting at the point $-3$ on the $x$ axis, proceed parallel to O$y$ until the point $P_1$ is reached corresponding to the level $-5$ on the $y$ axis. Then $P_1$ is marked and

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